APPLICATION OF GROUP SERIES TO SMALL ORDER GROUPS

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ABSTRACT: The aim of this research is to combine and upgrade the work of Ibrahim and Audu (2007) who give some basic procedure for computing wreath product of groups and that of Thanos (2006) who gives all the definitions related solvable groups and showed that any group of order up to 100 and not 60 is solvable. In this research, we apply group series to test the solvability and nilpotency status of small order groups, which began by constructing some groups of small order (Dihedral Groups and Groups Generated by Wreath Product), and test them for solvability and nilpotency using some hypothesis and theorems. The survey application of GAP are based on M. Bello et al (2017), where they constructed some groups, explore all their subgroups and test them for solvability. The standard program called Group Algorithms and Programming (GAP) is used to enhance and validate our result.

1.0 INTRODUCTION

Group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element. The operation satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility. One of the most familiar examples of a group is the set of integers together with the addition operation, but the abstract formalization of the group axioms, detached as it is from the concrete nature of any particular group and its operation, applies much more widely. It allows entities with highly diverse mathematical origins in abstract algebra and beyond to be handled in a flexible way while retaining their essential structural aspects. The ubiquity of groups in numerous areas within and outside mathematics makes them a central organizing principle of contemporary mathematics, Herstein (1975).

It may be worth though spending a few lines to mention how mathematicians came up with a concept of group. Around 1770, Lagrange initiated the study of permutations in connection with the study of the solution of equations. He was interested in understanding solutions of polynomials in several variables, and got this idea to study the behaviour of polynomials when their roots are permuted. This led to what we now call Lagrange's Theorem, though it was stated as If a function $f(x_1,\dots,x_n)$ of n variables is acted on by all n! Possible permutations of the variables and these permuted functions take on only r values, then r is a divisor of n!. It is Galois (1811-1832) who is considered by many as the founder of group theory. He was the first to use the term "group" in a technical sense, though to him it meant a collection of permutations closed under multiplication.

Galois was also motivated by the solvability of polynomial equations of degree n. From 1815 to 1844, Cauchy started to look at permutations as an autonomous subject, and introduced the concept of permutations generated by certain elements, as well as several notations still used today, such as the cyclic notation for permutations, the product of permutations, or the identity permutation. He proved what we call today Cauchy's Theorem, namely that if p is prime divisor of the cardinality of the group, then there exists a subgroup of cardinality p. In 1870, Jordan gathered all the applications of permutations he could find, from algebraic geometry, number theory, function theory, and gave a unified presentation (including the work of Cauchy and Galois). Jordan made explicit the notions of homomorphism, isomorphism (still for permutation groups); he introduced solvable groups, and proved that the indices in two composition series are the same (now called Jordan-Holder Theorem). He also gave a proof that the alternating group is simple for $n > 4$. In 1870, while working on number theory (more precisely, in generalizing Kummer's work on cyclotomic fields to arbitrary fields), Kronecker described in one of his papers a finite set of arbitrary elements on which he defined an abstract operation on them which satisfy certain laws, laws which now correspond to axioms for finite abelian groups. He used this definition to work with ideal classes. He also proved several results now known as theorems on abelian groups. Kronecker did not connect his definition with permutation groups, which was done in 1879 by Frobenius and Stickelberger.

1.2 DEFINITION OF TERMS

Group: A group is a non-empty set G on which there is a binary operation '*' such that;

- if a and b belong to G then $a * b$ is also in G (closure),
- $a * (b * c) = (a * b) * c$ for all a,b,c in G (associativity),
- there is an element $1 \in G$ such that $a * 1 = 1 * a = a$ for all $a \in G$ (identity),
- if $a \in G$, then there is an element $-a \in G$ such that $a * -a = -a * a = 1$ (inverse).

Subgroup: given a group G under a binary operation $*$, a subset H of G is called a subgroup of G if H also forms a group under the operation $*$. More precisely, H is a subgroup of G if the restriction of $*$ to $H \times H$ is a group operation on H

Subgroup Series: A subnormal series of a group G is a sequence of subgroups, each a normal subgroup of the next one. In a standard notation, there is no requirement made that A_i be a normal subgroup of G, only a normal subgroup of A_{i+1}. The quotient groups A_{i+1}/A_i are called the factor groups of the series.

Central Series: A series of subgroups $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset (e)$ is called a central series of a group G for all i, if $G_i \lhd G and\ G_i/G_{i+1} \subset Z(G/G_{i+1})$

Lower Central Series: Let $G_{(l)} = G$ and $G_{(i+l)} = gp(\{[g,x] | g \in G_{(i)}$and $x \in G\})$. The sequence of subgroups $G_{(1)} \supseteq G_{(2)} \supseteq \ldots \supset G_{(i)} \supset \ldots$is called thelower central series of G.

Composition Series: a **composition series** provides a way to break up an algebraic structure, such as a group or a module, into simple pieces i.e $\{\mathbf{1}\} = \boldsymbol{G_n} \rhd \boldsymbol{G_{n-1}} \rhd \cdots \rhd \boldsymbol{G_0} = \boldsymbol{G_i}$

Simple groups: A group $G \neq \{1\}$ is said to be simple if $\{1\}$ and G are the only normal subgroups of G

Dihedral group: A dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections.

Solvable Series: A group G is solvable if it has a subnormal series $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$ where each quotient G_i/G_{i+1} is an abelian group. We will call this a solvable series.

Solvable Group: A group G is said to be solvable if there exist a finite subnormal series for G such that each of its quotient group is abelian i.e. there exist a finite sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots G_n = (e)$ of subgroup of G.

Nilpotent Group: A group G is called nilpotent group of class r if it has a central series of length r. i.e. if $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_r = (e)$ is a central series of G.

Wreath Product: The Wreath product of C by D denoted by W = C wr D is the semidirect product of P by D, so that, $W = \{(f,d) \mid f \in P, d \in D\}$, with multiplication in W defined as $(f_1,d_1)(f_2,d_2) = ((f_1 f_2{}^{d_1{}^{-1}}), (d_1 d_2)$ for all $f_1, f_2 \in P and d_1, d_2 \in D$.

GROUP GENERATED BY WREATH PRODUCT

Recently, wreath product of groups has been used to explore some useful characteristics of finite groups in connection with permutation designes and construction of lattices Praeger and Scheider(2002), as well as in the study of interconnection networks, for instance. Further, Audu(2001) used wreath product to study the structure of some finite permutation groups. Wreath product constructions has been used to obtain for any positive integer n, solvable groups of derived length n, and commutator length at most equal to 2.

The Wreath product of C by D denoted by W = C wr D is the semidirect product of P by D, so that, $W = \{(f,d) \mid f \in P, d \in D\}$, with multiplication in W defined as $(f_1,d_1)(f_2,d_2) = ((f_1 f_2{}^{d_1{}^{-1}}), (d_1 d_2)$ for all $f_1, f_2 \in P and d_1, d_2 \in D$. Henceforth, we write f d instead of (f,d) for elements of W.

Theorem 1.1

Let D act on P as $f^d(\delta) = f(\delta d^{-1})$ where $f \in P, d \in D and \delta \in \Delta$. Let W be the group of all juxtaposed symbols f d, with $f \in P, d \in D$ and multiplication given by $(f_1, d_1)(f_2, d_2) = f_1 f_2^{d_1^{-1}}, (d_1 d_2)$. Then W is a group called the semi-direct product of P by D with the defined action.

Based on the forgoing we note the following:

- If C and D are finite groups, then the wreath product W determined by an action of D on a finite set is a finite group of order $|W| = |C|^{|\Delta|} . |D|$.
- P is a normal subgroup of W and D is a subgroup of W.
- The action of W on Γ × Δ is given by $(\alpha, \beta)fd = (\alpha f(\beta), \beta d) where \alpha \in \Gamma and \beta \in \Delta$.

We shall at this point identify the conditions under which a sup group will be soluble or nilpotent, and study them for further investigation.Audu (2007).

Theorem 1.2

If G is a group then the commutator subgroup G' is a normal subgroup of G and G/G' is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if N contains G'.

Proof

Let $f : G \to G$ be any automorphism of G. Then by the homomorphism property $f(aba^{-1}b^{-1}) = f(a)f(b)f(a^{-1})f(b^{-1}) = f(a)f(b)(f(a))^{-1}(f(b))^{-1} \in G'$. Then every element of G' is a finite product of powers of commutators $aba^{-1}b^{-1}$

(where $a, b \in G$) and so $f(G') < G'$. Let f_a be the automorphism of G given by conjugation by a. Then $aG'a^{-1} = f_a(G') < G'$. So every conjugate $aG'a^{-1}$is a subgroup of $G'$and then $G'$is a normal subgroup of G. Since all $a, b \in G$, we have $a^{-1}b^{-1} \in G$ and so $(a^{-1})^{-1}(b^{-1})^{-1} = a^{-1}b^{-1}ab \in G'$and so $a^{-1}b^{-1}abG'$ $= G'$ or $abG' = baG'$. But then by the definition of coset multiplication, $(aG')(bG') = abG' = baG' = (bG')(aG')$ and so coset multiplication is commutative and G/G' is abelian.

Theorem 1.3

A group G is solvable if and only if it has a solvable series.

Proof

Suppose G is solvable. Then by the definition of "solvable," in the derived series of commutator subgroups we have $G^{(n)} = (1)$, for some $n \in N$. By Theorem 2.2, in the series

$G > G^{(1)} > G^{(2)} > \cdots > G^{(2)} = (1)$, we have that$G^{(i+1)}$ is normal in $G^{(i)}$ and $G^{(i)}/G^{(i+1)}$ is abelian. So the series is subnormal (because each subgroup is normal in each previous subgroup) and is also solvable (since the quotient groups are abelian).

Now suppose $G = G_0 > G_1 > \cdots > G_n = (1)$ is a solvable series. Then

$G_i/G_{i+1}$is abelian (by definition of solvable series) for $0 \leq i \leq n-1$. By Theorem 2.2, $G_{i+1} > (G_i)'$for $0 \leq i \leq n-1$.

Since in the derived series of commutator subgroups we have $G > G^{(1)} > G^{(2)} > \cdots > G^{(n)}$, then $G_1 > G_0{}' = G' = G^{(1)}$

$$G_2 > G_1{}' = \left(G^{(1)}\right)' = G^{(2)}$$

$$G_3 > G_2{}' = \left(G^{(2)}\right)' = G^{(3)}$$

$$\vdots$$

$$G_{i+1} > G_i{}' = \left(G^{(i)}\right)' = G^{(i+1)}$$

$$\vdots$$

$$G_n > G'_{n-1} = \left(G^{(n-1)}\right)' = G^{(n)}.$$

But $G_n = (1)$ so it must be that $G^{(n)} = (1)$ and G is solvable.

Thus, we give the following illustrations:

(i) $G = \{(1), (48765), (47586), (46857), (45678), (132), (132)(48765), (132)(47586), (132)(46857), (132)(45678), (123), (123)(48765), (123)(47586), (123)(46857), (123)(45678)\}$

has the subgroups as follows;

$$H_0 = (1)$$

$$H_1 = \{(1), (123), (132)\}$$

$$H_2 = \{(1), (48765), (47586), (46857), (45678)\}$$

$$H_3 = \{(1), (48765), (47586), (46857), (45678), (132), (132)(48765), (132)(47586), (132)(46857), (132)(45678), (123), (123)(48765), (123)(47586), (123)(46857), (123)(45678)\}$$

has a solvable series which is $(1) = H_0 \lhd H_1 \lhd H_3 = G$ hence solvable by Theorem 1.3

(ii) The dihedral group D_n is solvable since $D_n \rhd \langle p \rangle \rhd \{1\}$
Let D_{16} be the Dihedral group of Degree 8 given by:

$$D_{16} = \{(1), (28)(37)(46), (15)(26)(37)(48), (15)(24)(68), (1753)(2864), (17)(26)(35), (1357)(2468), (13)(48)(57), (18765432), (18)(27)(36)(45), (14725836), (14)(23)(58)(67), (16385274), (16)(25)(34)(78), (12345678), (12)(38)(47)(56)\}$$

whose subgroups are as follows;

$$H_1 = (1)$$

$$H_2 = \{(1), (15)(26)(37)(48)\} = \langle p \rangle$$

$$H_3 = \{(1), (28)(37)(46), (15)(26)(37)(48), (15)(24)(68), (1753)(2864), (17)(26)(35), (1357)(2468), (13)(48)(57), (18765432), (18)(27)(36)(45), (14725836), (14)(23)(58)(67), (16385274), (16)(25)(34)(78), (12345678), (12)(38)(47)(56)\}$$

Hence $D_{16} = H_3 \rhd H_2 \rhd H_1 = (1)$

Proposition 1.3
Any group of order p^n where p is a prime, is solvable.
Proof
We prove the proposition by induction on n. For $n = 0$, the proposition is trivial. Let n ≥ 1 and assume that the proposition is true for $r < n$. Let G be a group of order p^n. Then by a Proposition, the centre C of G has order p^s where s ≥ 1. Then the order of G/C is p^{n-s} and $n - s < n$. By the induction hypothesis G/C is solvable.Milne (2013).

Theorem 1.4
Finite p-groups are nilpotent.
Proof
$Z_{r+1}(G)/Z_r(G) = Z(G/Z_r(G))$. Since the center of a non-trivial p-group is non-trivial, $Z_r(G) < Z_{r+1}(G)$ unless $Z_r(G) = G$.

Proposition 1.5
D_n is nilpotent if and only if $n = 2^i$ for some $i \geq 0$.

Proposition 1.6
D_{2n} is solvable for all $n \geq 1$.
Corollary 1.7

A group G is nilpotent if it has a central series.

Proof

If G is nilpotent then $\{e\} = Z_o(G) \subseteq Z_1(G) \subseteq \cdots \subseteq Z_n(G) = G$ is a central series of G.

Conversely, if $\{e\} = G_0 \subseteq \cdots \subseteq G_k = G$ is a central series of G then by the above hypothesis we have $G = G_k \subseteq Z_k(G),\ so\ G = Z_k(G)$, and so G is nilpotent.

Recall the commutator is given by $[x, y] = x^{-1}y^{-1}xy$.Baumslag and Chamdler (1968).

2.0 RESULTS

2.1 Consider the permutation groups $C_1 = \{(1), (123), (132)\}$ and $D_1 = \{(1), (12)\}$ acting on $X = \{1,2,3\}\ and\ \Delta = \{1,2\}$ respectively. Let $P_1 = {C_1}^{\Delta} = \{f: \Delta_1 \rightarrow C_1\}$. Then $|P_1| = |C_1|^{|\Delta_1|} = 3^2 = 9$. The order of the wreath product is given by $|W_1| = |C_1|^{|\Delta_1|} \times |D_1| = 3^2 \times 2$.

The mappings are as follows

$$f_1 : 1 \rightarrow (1), 2 \rightarrow (1)$$
$$f_2 : 1 \rightarrow (123), 2 \rightarrow (123)$$
$$f_3 : 1 \rightarrow (132), 2 \rightarrow (132)$$
$$f_4 : 1 \rightarrow (1), 2 \rightarrow (123)$$
$$f_5 : 1 \rightarrow (1), 2 \rightarrow (132)$$
$$f_6 : 1 \rightarrow (123), 2 \rightarrow (1)$$
$$f_7 : 1 \rightarrow (132), 2 \rightarrow (1)$$
$$f_8 : 1 \rightarrow (132), 2 \rightarrow (123)$$
$$f_9 : 1 \rightarrow (123), 2 \rightarrow (132)$$

The elements of W are

$$(f_1, d_1), (f_1, d_2), (f_2, d_1), (f_2, d_2), (f_3, d_1), (f_3, d_2), (f_4, d_1), (f_4, d_2), (f_5, d_1),$$
$$(f_5, d_2), (f_6, d_1), (f_6, d_2), (f_7, d_1), (f_7, d_2), (f_8, d_1), (f_8, d_2), (f_9, d_1), (f_9, d_2),$$
$$(\alpha, \delta)^{fd} = (\alpha f(\delta), \delta d).$$

Further, $\Gamma \times \Delta = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$
We obtain the following permutations by the action of W on $\Gamma \times \Delta$

$$(\alpha, \beta) f_1 d_1 = (\alpha f_i(\delta), \beta d_i)$$
$$(1,1) f_1 d_1 = (1 f_1(1), 1 d_1) = (1,1)$$
$$(1,2) f_1 d_1 = (1 f_1(2), 2 d_1) = (1,2)$$
$$(2,1) f_1 d_1 = (2 f_1(1), 1 d_1) = (2,1)$$
$$(2,2) f_1 d_1 = (2 f_1(2), 2 d_1) = (2,2)$$
$$(3,1) f_1 d_1 = (3 f_1(1), 1 d_1) = (3,1)$$
$$(3,2) f_1 d_1 = (3 f_1(2), 2 d_1) = (3,2)$$

Rename the symbols as

$(1,1) \rightarrow 1$	$(2,1) \rightarrow 2$	$(3,1) \rightarrow 3$
$(1,2) \rightarrow 4$	$(2,2) \rightarrow 5$	$(3,2) \rightarrow 6$

And in summary,

$$(\Gamma \times \Delta)^{(f_1, d_1)} = \begin{Bmatrix} (1,1)(1,2)(2,1)(2,2)(3,1)(3,2) \\ (1,1)(1,2)(2,1)(2,2)(3,1)(3,2) \end{Bmatrix}$$

$$(\Gamma\times\Delta)^{(f_1,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(1,2)(1,1)(2,2)(2,1)(3,2)(3,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_2,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,1)(2,2)(3,1)(3,2)(1,1)(1,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_2,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,2)(2,1)(3,2)(3,1)(1,2)(1,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_3,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,1)(3,2)(1,1)(1,2)(2,1)(2,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_3,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,2)(3,1)(1,2)(1,1)(2,2)(2,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_4,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(1,1)(2,2)(2,1)(3,2)(3,1)(1,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_4,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(1,2)(2,1)(2,2)(3,1)(3,2)(1,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_5,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(1,1)(3,2)(2,1)(1,2)(3,1)(2,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_5,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(1,2)(3,1)(2,2)(1,1)(3,2)(2,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_6,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,1)(1,2)(3,1)(2,2)(1,1)(3,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_6,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,2)(1,1)(3,2)(2,1)(1,2)(3,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_7,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,1)(1,2)(1,1)(2,2)(2,1)(3,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_7,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,2)(1,1)(1,2)(2,1)(2,2)(3,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_8,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,1)(2,2)(1,1)(3,2)(2,1)(1,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_8,d_2)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(3,2)(2,1)(1,2)(3,1)(2,2)(1,1)\end{Bmatrix}$$
$$(\Gamma\times\Delta)^{(f_9,d_1)}=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,1)(3,2)(3,1)(1,2)(1,1)(2,2)\end{Bmatrix}$$
$$(\Gamma\times\Delta)(f_9,d_2)=\begin{Bmatrix}(1,1)(1,2)(2,1)(2,2)(3,1)(3,2)\\(2,2)(3,1)(3,2)(1,1)(1,2)(2,1)\end{Bmatrix}$$

Then the permutations in cyclic form $3^2\times 2$ are

$$W_1=\begin{Bmatrix}(1),(14)(25)(36),(123)(456),(153426),(132)(465),\\(162435),(456),(142536),(465),(143625),(123),\\(152634),(132),(163524),(134)(456),(35),\\(123)(465),(15)(26)(34).\end{Bmatrix}$$

Some of the subgroups of $3^2\times 2$ are;

$$H_0=(1)$$
$$H_1=\{(1),(123),(132)\}$$
$$H_2=\{(1),(465),(456)(132),(132)(465),(132)(456),(123),(123)(465),(123)(456)\}$$
$$H_3=\begin{Bmatrix}(1),(14)(25)(36),(123)(456),(153426),(132)(465),\\(162435),(456),(142536),(465),(143625),(123),\\(152634),(132),(163524),(134)(456),(35),\\(123)(465),(15)(26)(34).\end{Bmatrix}$$

W is not nilpotent by Corollary 1.7 and hence solvable by Theorem 1.3 since it admits the solvable series $(1)=H_0\lhd H_1\lhd H_2\lhd H_3=W_1$

2.2 Consider the permutation groups $C_2 = \{(1),(12)\}$ and $D_2 = \{(1),(34)\}$ acting on $X = \{1,2\}$ *and* $\Delta = \{3,4\}$ respectively. Let $P = C^{\Delta} = \{f: \Delta \rightarrow C\}$. Then $|P| = |C|^{|\Delta|} = 2^2 = 4$. The order of the wreath product is given by $|W_2| = |C_2|^{|\Delta|} \times |D_2|$
Then the permutations in cyclic form are

$$W_2 = \{(1),(34),(12),(12)(34),(13)(24),(1324),(1423),(14)(23)\}$$
$$|W_2| = 2^2 \times 2 = 8$$
$$P_0 = (1)$$
$$P_1 = \{(1),(12)(34)\}$$
$$P_2 = \{(1),(34),(12),(12)(34),(13)(24),(1324),(1423),(14)(23)\}$$

W is nilpotent by Corollary 1.7 since it admits central series, $(1) = P_0 \subset P_1 \subset P_2 = W_2$ and hence solvable by Proposition 1.3.

2.3 Consider the permutation groups $C_3 = \{(1),(12)\}$ and $D_3 = \{(1),(123),(132)\}$ acting on $X = \{1,2,3\}$ *and* $\Delta = \{1,2\}$ respectively. Let $P = C^{\Delta} = \{f: \Delta \rightarrow C\}$. Then $|P| = |C|^{|\Delta|} = 2^3 = 8$. The order of the wreath product is given by $|W_3| = |C_3|^{|\Delta|} \times |D_3| = 2^3 \times 3$
Then the permutations in cyclic form are

$$W_3 = \left\{\begin{matrix} (1),(56)(34),(34),(56),(12),(12)(56),(12)(34),(12)(34)(56), \\ (153)(264),(154263),(153264),(154)(263),(164253),(163)(254),(164) \\ (253),(163254),(135)(246),(135246),(136245),(136)(245),(146235), \\ (146)(235),(145)(236),(145236) \end{matrix}\right\}$$
$$W_3 = 2^3 \times 3 = 24$$
$$T_1 = (1)$$
$$T_2 = \{(1),(34)\}$$
$T_3 = \{(1),(12)(56),(12)(56)(34),(12)(34)(34)(56),(12)(34)(56)\}$
$$T_4 = \left\{\begin{matrix} (1),(56)(34),(34),(56),(12),(12)(56),(12)(34),(12)(34)(56), \\ (153)(264),(154263),(153264),(154)(263),(164253),(163)(254),(164) \\ (253),(163254),(135)(246),(135246),(136245),(136)(245),(146235), \\ (146)(235),(145)(236),(145236) \end{matrix}\right\}$$

W is not nilpotent by Corollary 1.7 and hence solvable by Theorem 1.3 since it admits the solvable series $(1) = T_1 \lhd T_2 \lhd T_3 \lhd T_4 = W_3$

2.4 Consider the permutation groups $C_4 = \{(1),(123),(132)\}$ and $D_4 = \{(1),(456),(465)\}$ acting on $X = \{1,2,3\}$ *and* $\Delta = \{4,5,6\}$ respectively. Let $P = C^{\Delta} = \{f: \Delta \rightarrow C\}$. Then $|P| = |C|^{|\Delta|} = 3^3 = 27$.The order of the wreath product is given by $|W_4| = |C_4|^{|\Delta|} \times |D_4|$
Then the permutations in cyclic form are

$$W_4 = \left\{\begin{matrix} (1),(798),(789),(465),(465)(798),(465)(789),(456),(456)(798),(456)(789),(132),(132) \\ (798),(132)(789),(132)(465),(132)(465)(798),(132)(465)(789),(132)(456),(132)(456) \\ (798),(132)(456)(789),(123),(123)(798),(123)(789),(123)(465),(123)(465)(798),(123) \\ (465)(789),(123)(456),(123)(456)(798),(123)(456)(789),(174)(285)(396),(176395284), \\ (175286394),(174396285),(176284395),(175)(286)(394),(174285396),(176)(284)(395), \\ (175394286),(196385274),(195276384),(194)(275)(386),(196274385),(195)(276)(384), \\ (194386275),(196)(274)(385),(195384276),(194275386),(185296374),(184)(295)(376), \\ (186375294),(185)(296)(374),(184376295),(186294375),(185374296),(184295376), \\ (186)(294)(375),(147)(258)(369),(147369258),(147258369),(149368257),(149257368), \\ (149)(257)(368),(148259367),(148)(259)(367),(148367259),(169358247),(169247358), \\ (169)(247)(358),(168249357),(168)(249)(357),(168357249),(167)(248)(359), \\ (167359248),(167248359),(158269347),(158)(269)(347),(158347269),(157)(268) \\ (349),(157349268),(157268349),(159348267),(159267348),(159)(267)(348) \end{matrix}\right\}$$

$$|W_4| = 3^3 \times 3$$

$P_0 = (1)$, $P_1 = \{(1),(123)(456)(789)\}$, $P_2 = \{(465)(789),(123)(456)(789)\}$

$$P_3 = \left\{ \begin{array}{c}
(1), (798), (789), (465), (465)(798), (465)(789), (456), (456)(798), (456)(789), (132), \\
(132)(798), (132)(789), (132)(465), (132)(465)(798), (132)(465)(789), (132)(456), (132) \\
(456)(798), (132)(456)(789), (123), (123)(798), (123)(789), (123)(465), (123)(465) \\
(798), (123)(465)(789), (123)(456), (123)(456)(798), (123)(456)(789), (174)(285) \\
(396), (176395284), (175286394), (174396285), (176284395), (175)(286)(394), \\
(174285396), (176)(284)(395), (175394286), (196385274), (195276384), (194)(275) \\
(386), (196274385), (195)(276)(384), (194386275), (196)(274)(385), (195384276), \\
(194275386), (185296374), (184)(295)(376), (186375294), (185)(296)(374), (18437 \\
6295), (186294375), (185374296), (184295376), (186)(294)(375), (147)(258)(369), \\
(147369258), (147258369), (149368257), (149257368), (149)(257)(368), (148259367), \\
(148)(259)(367), (148367259), (169358247), (169247358), (169)(247)(358), \\
(168249357), (168)(249)(357), (168357249), (167)(248)(359), (167359248), \\
(167248359), (158269347), (158)(269)(347), (158347269), (157)(268)(349), \\
(157349268), (157268349), (159348267), (159267348), (159)(267)(348)
\end{array} \right\}$$

W_4 is nilpotent by Corollary 1.7 since it admits central series, $(1) = P_0 \subset P_1 \subset P_2 \subset P_3 = W_4$ and hence solvable by Proposition 1.3

2.5 Consider the permutation groups $C_5 = \{(1), (12)\}$ and $D_5 = \{(1), (34567), (35746), (36475), (37654)\}$ acting on $X = \{1,2\}$ *and* $\Delta = \{3,4,5,6,7\}$ respectively. Let $P = C^{\Delta} = \{f: \Delta \to C\}$. Then $|P| = |C|^{|\Delta|} = 2^5 = 32$. The order of the wreath product is given by $|W_5| = |C_5|^{|\Delta|} \times |D_5|$
Then the permutations in cyclic form are

$$W = \left\{ \begin{array}{c}
(1), (9\ 10), (78), (78)(9\ 10), (56), (56)(9\ 10), (56)(78), (56)(78)(9\ 10), (34), (34) \\
(9\ 10), (34)(78), (34)(78)(9\ 10), (34)(56), (34)(56)(9\ 10), (34)(56)(78), (34)(56)(78) \\
(9\ 10), (12), (12)(9\ 10), (12)(78), (12)(78)(9\ 10), (12)(56), (12)(56)(9\ 10), (12)(56)(78), \\
(12)(56)(78)(9\ 10), (12)(34), (12)(34)(9\ 10), (12)(34)(78), (12)(34)(78)(9\ 10), (12)(34) \\
(56), (12)(34)(56)(9\ 10), (12)(34)(56)(78), (12)(34)(56)(78)(9\ 10), (19753)(2\ 10\ 864), \\
(198642\ 10\ 753), (197642\ 10\ 853), (19853)(2\ 10\ 764), (197542\ 10\ 863), (19863)(2\ 10\ 754), \\
(19763)(2\ 10\ 854), (198542\ \ 10\ \ 763), (197532\ 10\ 864), (19864)(2\ 10\ 753), (19764)(2\ 10\ 853), \\
(198532\ 10\ 764), (19754)(2\ 10\ 863), (198632\ 10\ 754), (197632\ 10\ 854), (19854)(2\ 10\ 763), \\
(1\ 10\ 86429753), (1\ 10\ 753)(29864), (1\ 10\ 853)(29764), (1\ 10\ 76429853), (1\ 10\ 863)(29754), \\
(1\ 10\ 75429863), (1\ 10\ 85429763), (1\ 10\ 763)(29854), (1\ 10\ 864)(29753), (1\ 10\ 75329864), \\
(1\ 10\ 85329764), (1\ 10\ 764)(29853), (1\ 10\ 86329754), (1\ 10\ 754)(29863), (1\ 10\ 854)(29763), \\
(1\ 10\ 76329854), (17395)(284\ 10\ 6), (17396284\ 10\ 5), (174\ 10\ 628395), (174\ 10\ 5)(28396), \\
(17395284\ 10\ 6), (17396)(284\ 10\ 5), (174\ 10\ 6)(28395), (174\ 10\ 528396), (173\ 10\ 628495), \\
(173\ 10\ 5)(28496), (17495)(283\ 10\ 6), (17496283\ 10\ 5), (173\ 10\ 6)(28495), (173\ 10\ 528496), \\
(17495283\ 10\ 6), (17496)(283\ 10\ 5), (184\ 10\ 627395), (184\ 10\ 5)(27396), (18395)(274\ 10\ 6), \\
(18396274\ 10\ 5), (184\ 10\ 6)(27395), (184\ 10\ 527396), (18395274\ 10\ 6), (18396)(274\ 10\ 5), \\
(18495)(273\ 10\ 6), (18496273\ 10\ 5), (183\ 10\ 627495), (183\ 10\ 5)(27496), (18495273\ 10\ 6), \\
(18496)(273\ 10\ 5), (183\ 10\ 6)(27495), (183\ 10\ 527496), (15937)(26\ 10\ 48), (1594826\ 10\ 37), \\
(1593726\ 10\ 48), (15948)(26\ 10\ 37), (15\ 10\ 4826937), (15\ 10\ 37)(26948), (15\ 10\ 48)(26937), \\
(15\ 10\ 3726948), (1593826\ 10\ 47), (15947)(26\ 10\ 38), (15938)(26\ 10\ 47), (1594726\ 10\ 38), \\
(15\ 10\ 47)(26938), (15\ 10\ 3826947), (15\ 10\ 4726938), (15\ 10\ 38)(26947), (16\ 10\ 4825937), \\
(16\ 10\ 37)(25948), (16\ 10\ 48)(25937), (16\ 10\ 3725948), (16937)(25\ 10\ 48), (1694825\ 10\ 37), \\
(1693725\ 10\ 48), (16948)(25\ 10\ 37), (16\ 10\ 47)(25938), (16\ 10\ 3825947), (16\ 10\ 4725938), \\
(16\ 10\ 38)(25947), (1693825\ 10\ 47), (16947)(25\ 10\ 38), (16938)(25\ 10\ 47), (1694725\ 10\ 38), \\
(13579)(2468\ 10), (135792468\ 10), (1357\ 10\ 24689), (1357\ 10)(24689), (1358\ 10\ 24679), \\
(1358\ 10)(24679), (13589)(2467\ 10), (135892467\ 10), (1368\ 10\ 24579), (1368\ 10)(24579), \\
(13689)(2457\ 10), (136892457\ 10), (13679)(2458\ 10), (136792458\ 10), (1367\ 10\ 24589), \\
(1367\ 10)(24589), (1468\ 10\ 23579), (1468\ 10)(23579), (14689)(2357\ 10), (146892357\ 10), \\
(14679)(2358\ 10), (146792358\ 10), (1467\ 10\ 23589), (1467\ 10)(23589), (14579)(2368\ 10), \\
(145792368\ 10), (1457\ 10\ 23689), (1457\ 10)(23689), (1458\ 10\ 23679), (1458\ 10)(23679), \\
(14589)(2367\ 10), (145892367\ 10)
\end{array} \right.$$

$$|W_5| = 2^5 \times 5 = 160$$

$$H_0 = (1)$$
$$H_1 = \{(1), (56), (65)\}$$
$$H_2 = \{(5,6), (7,8), (7,8)(9,10), (1,2)(9,10), (3,4)\}$$

$$H_3 = \left\{ \begin{array}{c}
(1), (9\ 10\ ), (78), (78)(9\ 10\ ), (56), (56)(9\ 10\ ), (56)(78), (56)(78)(9\ 10\ ), (34), (34) \\
(9\ 10\ ), (34)(78), (34)(78)(9\ 10\ ), (34)(56), (34)(56)(9\ 10\ ), (34)(56)(78), (34)(56)(78) \\
(9\ 10\ ), (12), (12)(9\ 10\ ), (12)(78), (12)(78)(9\ 10\ ), (12)(56), (12)(56)(9\ 10\ ), (12)(56)(78), \\
(12)(56)(78)(9\ 10\ ), (12)(34), (12)(34)(9\ 10\ ), (12)(34)(78), (12)(34)(78)(9\ 10\ ), (12)(34) \\
(56), (12)(34)(56)(9\ 10\ ), (12)(34)(56)(78), (12)(34)(56)(78)(9\ 10\ ), (19753)(2\ 10\ 864), \\
(198642\ 10\ 753), (197642\ 10\ 853), (19853)(2\ 10\ 764), (197542\ 10\ 863), (19863)(2\ 10\ 754), \\
(19763)(2\ 10\ 854), (198542\ \ 10\ \ 763), (197532\ 10\ 864), (19864)(2\ 10\ 753), (19764)(2\ 10\ 853), \\
(198532\ 10\ 764), (19754)(2\ 10\ 863), (198632\ 10\ 754), (197632\ 10\ 854), (19854)(2\ 10\ 763), \\
(1\ 10\ 86429753), (1\ 10\ 753)(29864), (1\ 10\ 853)(29764), (1\ 10\ 76429853), (1\ 10\ 863)(29754), \\
(1\ 10\ 75429863), (1\ 10\ 85429763), (1\ 10\ 763)(29854), (1\ 10\ 864)(29753), (1\ 10\ 75329864), \\
(1\ 10\ 85329764), (1\ 10\ 764)(29853), (1\ 10\ 86329754), (1\ 10\ 754)(29863), (1\ 10\ 854)(29763), \\
(1\ 10\ 76329854), (17395)(284\ 10\ 6), (17396284\ 10\ 5), (174\ 10\ 628395), (174\ 10\ 5)(28396), \\
(17395284\ 10\ 6), (17396)(284\ 10\ 5), (174\ 10\ 6)(28395), (174\ 10\ 528396), (173\ 10\ 628495), \\
(173\ 10\ 5)(28496), (17495)(283\ 10\ 6), (17496283\ 10\ 5), (173\ 10\ 6)(28495), (173\ 10\ 528496), \\
(17495283\ 10\ 6), (17496)(283\ 10\ 5), (184\ 10\ 627395), (184\ 10\ 5)(27396), (18395)(274\ 10\ 6), \\
(18396274\ 10\ 5), (184\ 10\ 6)(27395), (184\ 10\ 527396), (18395274\ 10\ 6), (18396)(274\ 10\ 5), \\
(18495)(273\ 10\ 6), (18496273\ 10\ 5), (183\ 10\ 627495), (183\ 10\ 5)(27496), (18495273\ 10\ 6), \\
(18496)(273\ 10\ 5), (183\ 10\ 6)(27495), (183\ 10\ 527496), (15937)(26\ 10\ 48), (1594826\ 10\ 37), \\
(1593726\ 10\ 48), (15948)(26\ 10\ 37), (15\ 10\ 4826937), (15\ 10\ 37)(26948), (15\ 10\ 48)(26937), \\
(15\ 10\ 3726948), (1593826\ 10\ 47), (15947)(26\ 10\ 38), (15938)(26\ 10\ 47), (1594726\ 10\ 38), \\
(15\ 10\ 47)(26938), (15\ 10\ 3826947), (15\ 10\ 4726938), (15\ 10\ 38)(26947), (16\ 10\ 4825937), \\
(16\ 10\ 37)(25948), (16\ 10\ 48)(25937), (16\ 10\ 3725948), (16937)(25\ 10\ 48), (1694825\ 10\ 37), \\
(1693725\ 10\ 48), (16948)(25\ 10\ 37), (16\ 10\ 47)(25938), (16\ 10\ 3825947), (16\ 10\ 4725938), \\
(16\ 10\ 38)(25947), (1693825\ 10\ 47), (16947)(25\ 10\ 38), (16938)(25\ 10\ 47), (1694725\ 10\ 38), \\
(13579)(2468\ 10\ ), (135792468\ 10\ ), (1357\ 10\ 24689), (1357\ 10\ )(24689), (1358\ 10\ 24679), \\
(1358\ 10\ )(24679), (13589)(2467\ 10\ ), (135892467\ 10\ ), (1368\ 10\ 24579), (1368\ 10\ )(24579), \\
(13689)(2457\ 10\ ), (136892457\ 10\ ), (13679)(2458\ 10\ ), (136792458\ 10\ ), (1367\ 10\ 24589), \\
(1367\ 10\ )(24589), (1468\ 10\ 23579), (1468\ 10\ )(23579), (14689)(2357\ 10\ ), (146892357\ 10\ ), \\
(14679)(2358\ 10\ ), (146792358\ 10\ ), (1467\ 10\ 23589), (1467\ 10\ )(23589), (14579)(2368\ 10\ ), \\
(145792368\ 10\ ), (1457\ 10\ 23689), (1457\ 10\ )(23689), (1458\ 10\ 23679), (1458\ 10\ )(23679), \\
(14589)(2367\ 10\ ), (145892367\ 10\ )
\end{array} \right\}$$

W_5 is not nilpotent by Corollary 1.7 and hence solvable by Theorem 1.3 since it admits the solvable series $(1) = H_0 \lhd H_1 \lhd H_2 \lhd H_3 = W_5$

2.6 DIHEDRAL GROUPS

2.6.1 For $n = 3$

$D_6 = \{(1), (23), (132), (13), (123), (12)\}$

Some subgroups of D_6 are as follows;

$$H_1 = (1)$$
$$H_2 = \{(1), (123), (132)\}$$
$$H_3 = \{(1), (23), (132), (13), (123), (12)\}$$

D_6 is not nilpotent by Corollary 1.7 and hence solvable by Theorem 1.3 since it admits the solvable series $(1) = H_1 \lhd H_2 \lhd H_3 = D_6$

2.6.2 For $n = 4$

$D_8 = \{(1), (24), (13)(24), (13), (1432), (14)(23), (1234), (12)(34)\}$

Some subgroups of D_8 are as follows;

$$H_1 = (1)$$
$$H_2 = \{(1), (13)(24)\}$$
$$H_3 = \{(1), (24), (13)(24), (13), (1432), (14)(23), (1234), (12)(34)\}$$

D_8 is nilpotent by Corollary 1.7 since it admits central series, $(1) = H_1 \subset H_2 \subset H_3 = D_8$ and hence solvable by Proposition 1.3

2.6.3 For $n = 12$

$D_{24} = \{(1), (\ 2\ 12)(\ 311)(\ 410)(\ 5\ 9)(\ 6\ 8), (\ 1\ 9\ 5)(\ 210\ 6)(\ 311\ 7)(\ 412\ 8),$
$(\ 1\ 9)(\ 2\ 8)(\ 3\ 7)(\ 4\ 6)(1012), (\ 1\ 5\ 9)(\ 2\ 610)(\ 3\ 711)(\ 4\ 812), (\ 1\ 5)(\ 2\ 4)(\ 612)(\ 711)(\ 810),$
$(\ 111\ 9\ 7\ 5\ 3)(\ 21210\ 8\ 6\ 4), (\ 111)(\ 210)(\ 3\ 9)(\ 4\ 8)(\ 5\ 7), (\ 1\ 7)(\ 2\ 8)(\ 3\ 9)(\ 410)(\ 511)(\ 612),$
$(\ 1\ 7)(\ 2\ 6)(\ 3\ 5)(\ 812)(\ 911), (\ 1\ 3\ 5\ 7\ 911)(\ 2\ 4\ 6\ 81012), (\ 1\ 3)(\ 412)(\ 511)(\ 610)(\ 7\ 9),$
$(\ 1121110\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2), (\ 112)(\ 211)(\ 310)(\ 4\ 9)(\ 5\ 8)(\ 6\ 7), (\ 1\ 8\ 310\ 512\ 7\ 2\ 9\ 411\ 6),$
$(\ 1\ 8)(\ 2\ 7)(\ 3\ 6)(\ 45)(\ 912)(1011), (\ 1\ 4\ 710)(\ 2\ 5\ 811)(\ 3\ 6\ 912)$
$, (\ 1\ 4)(\ 2\ 3)(\ 512)(\ 611)(\ 710)(\ 8\ 9), (\ 110\ 7\ 4)(\ 211\ 8\ 5)$
$(\ 312\ 9\ 6), (\ 110)(\ 2\ 9)(\ 3\ 8)(\ 4\ 7)(\ 5\ 6)(1112), (\ 1\ 611\ 4\ 9\ 2\ 712\ 510\ 3\ 8), (\ 1\ 6)(\ 2\ 5)(\ 3\ 4)$
$(\ 712)(\ 811)(\ 910), (\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9101112), (\ 1\ 2)(\ 312)(\ 411)(\ 510)(\ 6\ 9)(\ 7\ 8)\}$

Some subgroups of D_{24} are as follows;

$$H_1 = (1)$$

$H_2 = \{(1), (\ 1\ 9\ 5)(\ 210\ 6)(\ 311\ 7)(\ 412\ 8), (\ 1\ 5\ 9)(\ 2\ 610)(\ 3\ 711)(\ 4\ 812),$
$(\ 111\ 9\ 7\ 5\ 3)(\ 21210\ 8\ 6\ 4), (\ 1\ 7)(\ 2\ 8)(\ 3\ 9)(\ 410)(\ 511)(\ 612), (\ 1\ 3\ 5\ 7\ 911)(\ 2\ 4\ 6\ 81012),$
$(\ 1121110\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2), (\ 1\ 8\ 310\ 512\ 7\ 2\ 9\ 411\ 6), (\ 1\ 4\ 710)(\ 2\ 5\ 811)(\ 3\ 6\ 912),$
$(\ 110\ 7\ 4)(\ 211\ 8\ 5)(\ 312\ 9\ 6), (\ 1\ 611\ 4\ 9\ 2\ 712\ 510\ 3\ 8), (\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9101112)\}$

$H_3 = \{(1), (\ 2\ 12)(\ 311)(\ 410)(\ 5\ 9)(\ 6\ 8), (\ 1\ 9\ 5)(\ 210\ 6)(\ 311\ 7)(\ 412\ 8),$
$(\ 1\ 9)(\ 2\ 8)(\ 3\ 7)(\ 4\ 6)(1012), (\ 1\ 5\ 9)(\ 2\ 610)(\ 3\ 711)(\ 4\ 812), (\ 1\ 5)(\ 2\ 4)(\ 612)(\ 711)(\ 810),$
$(\ 111\ 9\ 7\ 5\ 3)(\ 21210\ 8\ 6\ 4), (\ 111)(\ 210)(\ 3\ 9)(\ 4\ 8)(\ 5\ 7), (\ 1\ 7)(\ 2\ 8)(\ 3\ 9)(\ 410)(\ 511)(\ 612),$
$(\ 1\ 7)(\ 2\ 6)(\ 3\ 5)(\ 812)(\ 911), (\ 1\ 3\ 5\ 7\ 911)(\ 2\ 4\ 6\ 81012), (\ 1\ 3)(\ 412)(\ 511)(\ 610)(\ 7\ 9),$
$(\ 1121110\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2), (\ 112)(\ 211)(\ 310)(\ 4\ 9)(\ 5\ 8)(\ 6\ 7), (\ 1\ 8\ 310\ 512\ 7\ 2\ 9\ 411\ 6),$
$(\ 1\ 8)(\ 2\ 7)(\ 3\ 6)(\ 45)(\ 912)(1011), (\ 1\ 4\ 710)(\ 2\ 5\ 811)(\ 3\ 6\ 912)$
$, (\ 1\ 4)(\ 2\ 3)(\ 512)(\ 611)(\ 710)(\ 8\ 9), (\ 110\ 7\ 4)(\ 211\ 8\ 5)$
$(\ 312\ 9\ 6), (\ 110)(\ 2\ 9)(\ 3\ 8)(\ 4\ 7)(\ 5\ 6)(1112), (\ 1\ 611\ 4\ 9\ 2\ 712\ 510\ 3\ 8), (\ 1\ 6)(\ 2\ 5)(\ 3\ 4)$
$(\ 712)(\ 811)(\ 910), (\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9101112), (\ 1\ 2)(\ 312)(\ 411)(\ 510)(\ 6\ 9)(\ 7\ 8)\}$

D_{24} is not nilpotent by Corollary 1.7 and hence solvable by Theorem 1.3 since it admits the solvable series $(1) = H_0 \lhd H_1 \lhd H_2 \lhd H_3 = D_{24}$

TABLE 4.1: RESULT SUMMARY

Method/Type	Order	Status	
		Solvable	Nilpotent
Wreath Product	$3^2 \times 2$	T	F
	$2^2 \times 2$	T	T
	$2^3 \times 3$	T	F
	$3^3 \times 3$	T	T
	$2^5 \times 5$	T	F
Dihedral Group	2×3	T	F
	2^3	T	T
	$2^3 \times 3$	T	F

Key: ***True = T False = F***

SUMMARY

In this research work we were able to construct dihedral groups and groups generated by wreath product, the construction involves groups of order $pq,\ p^r q$ and p^k where p, q are prime and k is integer. It is noticed that if the order of group is $pq,\ P^r q \ where\ P \neq q$ is solvable but not nilpotent whereas if $p = q$ that is a group of order P^k, then the group is both solvable and nilpotent.

CONCLUSION

In conclusion, we have been able to construct some group of small order (Dihedral groups and Groups generated by wreath products of two permutations) and also find the groups order, the status of the group in terms of Solvability and Nilpotency, draft out the subgroup series and hence found out that there is a consistent in the group properties in terms of solvability and nilpotency between dihedral groups and groups generated by wreath product.

RECOMMENDATION

We recommended that for further research, symmetric groups should also be included and tested for the same properties.

3.0 VALIDATION OF RESULTS

3.1 Algorithms for the result in 2.1

```
gap> C1:=Group((1,2,3));
Group([ (1,2,3) ])
gap> D1:=Group((1,2));
Group([ (1,2) ])
gap> W1:=WreathProduct(C1,D1);
Group([ (1,2,3), (4,5,6), (1,4)(2,5)(3,6) ])
gap> H:=SubnormalSeries(W,Group([(1,5)(2,6)(3,4)]));
[ Group([ (1,2,3), (4,5,6), (1,4)(2,5)(3,6) ]), Group([
(1,5)(2,6)(3,4), (1,4)(2,5)(3,6) ]) ]
gap> Order(W1);
18
gap>IsSolvable(W1);
true
gap>IsNilpotent(W1);
false
gap> quit;
```

3.2 Algorithms for the result in 2.2

```
gap> C2:=Group((1,2));
Group([ (1,2) ])
gap> D2:=Group((3,4));
Group([ (3,4) ])
gap> W2:=WreathProduct(X,D);
Group([ (1,2), (3,4), (1,3)(2,4) ])
gap> L2:=LowerCentralSeriesOfGroup(W2);
[ Group([ (1,2), (3,4), (1,3)(2,4) ]), Group([ (1,2)(3,4) ]),
Group(()) ]
gap> Order(W2);
8
gap>IsSolvable(W2);
true
gap>IsNilpotent(W2);
true
gap> quit;
```

3.3 Algorithms for the result in 2.3

```
gap> C3:=Group((1,2));
Group([ (1,2) ])
gap> D3:=Group((3,4,5));
Group([ (3,4,5) ])
gap> W3:=WreathProduct(C3,D3);
Group([ (1,2), (3,4), (5,6), (1,3,5)(2,4,6) ])
gap> T3:=SubnormalSeries(W3,Group([(1,2)]));
[ Group([ (1,2,3), (4,5,6), (1,4)(2,5)(3,6) ]), Group([
(1,2), (2,3), (4,5), (5,6) ]), Group([ (1,2), (1,3) ]) ]
gap> Order(W3);
24
gap>IsSolvable(W3);
true
gap>IsNilpotent(W3);
false
gap> quit;
```

3.4 Algorithms for the result in 2.4

```
gap> C4:=Group((1,2,3));
Group([ (1,2,3) ])
gap> D4:=Group((4,5,6));
Group([ (4,5,6) ])
gap> W4:=WreathProduct(C4,D4);
Group([ (1,2,3), (4,5,6), (7,8,9), (1,4,7)(2,5,8)(3,6,9) ])
gap> P:=LowerCentralSeriesOfGroup(W);
[ Group([ (1,2,3), (4,5,6), (7,8,9), (1,4,7)(2,5,8)(3,6,9) ]),
Group([ (4,6,5)(7,8,9), (1,2,3)(4,5,6)(7,8,9) ]), Group([
(1,2,3)(4,5,6)(7,8,9) ]), Group(()) ]
gap> Order(W4);
81
gap>IsSolvable(W4);
true
gap>IsNilpotent(W4);
true
gap> quit;
```

3.5 Algorithms for the result in 2.5

```
gap> C5:=Group((1,2));
Group([ (1,2) ])
gap> D5:=Group((3,4,5,6,7));
Group([ (3,4,5,6,7) ])
gap> W5:=WreathProduct(C5,D5);
Group([ (1,2), (3,4), (5,6), (7,8), (9,10),
(1,3,5,7,9)(2,4,6,8,10) ])
gap>
H:=SubnormalSeries(W,Group([(1,3,5,7,10)(2,4,6,8,9)])
);
[ Group([ (1,2), (3,4), (5,6), (7,8), (9,10),
(1,3,5,7,9)(2,4,6,8,10) ]), Group([
(1,3,5,7,10)(2,4,6,8,9), (1,3,5,7,9)(2,4,6,8,10) ]) ]
gap> Order(W5);
160
gap>IsSolvable(W5);
true
gap>IsNilpotent(W5);
false
gap> quit;
```

3.6 Algorithms for the result in 2.6.1

```
gap> D6:=DihedralGroup(IsGroup, 6);
Group([ (1,2,3), (2,3) ])
gap> H:=SubnormalSeries(D6,Group([(1,2,3)]));
[ Group([ (1,2,3), (2,3) ]), Group([ (1,2,3) ]) ]
gap>IsSolvable(D6);
true
gap>IsNilpotent(D6);
false
gap> quit;
```

3.7 Algorithms for the result in 3.6.2

```
gap> D8:=DihedralGroup(IsGroup, 8);
Group([ (1,2,3,4), (2,4) ])
gap> P:=LowerCentralSeriesOfGroup(D8);
[ Group([ (1,2,3,4), (2,4) ]), Group([ (1,3)(2,4) ]),
Group(()) ]
gap> Order(D8);
8
gap>IsSolvable(D8);
true
```

```
gap>IsNilpotent(D8);
true
gap> quit;
```

3.8 Algorithms for the result in 3.6.3

```
gap> D10:=DihedralGroup(IsGroup, 10);
Group([ (1,2,3,4,5), (2,5)(3,4) ])
gap> Q:=SubnormalSeries(D10,Group([(2,5)(3,4)]));
[ Group([ (1,2,3,4,5), (2,5)(3,4) ]) ]
gap>IsSolvable(D10);
true
gap>IsNilpotent(D10);
false
gap> quit;
```

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